Physico-statistical model of rainfall flood formation and determination of its parameters

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Abstract. The transfer from the rainfall runoff model on an elementary plot to the model describing the rainfall runoff formation in the catchment is achieved by theoretical probability averaging. An optimization method is applied to determine the model parameters. The model is tested on the observed data of a few small catchments.

Résumé. On utilise les méthodes probabilistes de calcul des moyennes pour réalisez le passage du modèle de formation de l'écoulement des pluies sur une surface élémentaire au modèle décrivant la formation de l'écoulement des pluies sur le bassin. Pour déterminer les paramètres du modèle on utilise les méthodes d'optimisation avec complexisation altéreure du modèle. Le modèle est verifie d'après diverses observation sur plusieurs bassins peu étendus.

It is not possible to separate the selection of model structure from the determination of the model parameters in the construction of a mathematical rainfall runoff model designed for use in hydrological forecasting and calculations.

A complex model structure allows widely different runoff conditions to be described and is more reliable in the cases which have not been used for the model calibration (parameter determination). However, due to the lack of sufficient data this results in estimated parameters of reduced accuracy. On the other hand, for the purposes of parameter determination, it is desirable to obtain a model of minimum complexity but in such a model if the diversity of the runoff conditions is neglected the parameter estimates will be unstable.

Obviously, the model of optimum complexity can be obtained only by accounting for the diverse runoff formation conditions and by considering the data available on each specific catchment. Therefore, it is more reasonable to speak not about the construction of the runoff formation model in general but about the runoff formation model for a specific catchment. Consequently we can recommend a general model structure from which an individual structure could be singled out for each specific catchment. The object of hydrological theory is to narrow, as far as possible, the class to which the general model structure is assigned.

The search for such a narrow class of models has resulted in the model where the runoff is considered as a dynamic system with lumped parameters. This direction seems to be the most fruitful if one considers the available data and the basic hydrological problems where, as a rule, it is required merely to calculate a hydrograph at the outlet (without detailing the water regime of the catchment proper). However, consideration of the particular catchment as a system with lumped parameters makes it difficult to use physical laws and relations which are known from experimental hydrology. Model construction is often empirical: a search is made for a system of ordinary differential equations or any corresponding operator which matches the measured hydrometeorological effects on the catchment (input values) with their corresponding runoff hydrographs ('black box' method). Such models can yield good results for many problems though their physical interpretation is rather difficult. This manifests itself especially when passing to catchments on which there are insufficient data available.

One way of constructing physically and experimentally based models of runoff formation with lumped parameters is based on the transfer by probability averaging from a description of the processes occurring on an elementary plot to a description of the processes in the catchment. Such an approach has already been applied to the construction of the snowmelt runoff formation (Komarov, 1959; Popov, 1963). It is now used to construct a model of the rainfall−runoff formation.

The water yield rate of elementary plot i is

$$q_i(t) = P_i(t) - E_i(t) - I_i(t) \tag{1}$$

where $P_i(t)$ is the precipitation rate, $E_i(t)$ is the evaporation rate, and $I_i(t)$ is the infiltration rate.

The evaporation and infiltration rates can be determined from the following relations (assuming that the calculated time interval is great compared to the recession period of infiltration rate):

$$E_i(t) = \begin{cases} k_1[1 + k_2 U_i(t)] D_i(t) & \text{if} \quad d_i < W_m \\ 0 & \text{if} \quad d_i \geqslant W_m \end{cases} \tag{2}$$

$$I_i(t) = \begin{cases} \dfrac{d_i(t)}{k_3} + i_0 & \text{if} \quad P_i(t) - E_i(t) > I_i(t) \\ P_i(t) - E_i(t) & \text{if} \quad P_i(t) - E_i(t) \leqslant I_i(t) \end{cases} \tag{3}$$

where

k_1, k_2, k_3 and i_0 are empirical parameters which are assumed to be constant for the entire catchment;

$U_i(t)$ is the wind velocity;

$D_i(t)$ is the air moisture deficit;

W_m is the maximum soil moisture storage capacity;

d_i is the soil moisture deficit.

The mathematical water yield rate of the entire catchment is expected to be equal to:

$$M[q] = M[P] - M[E] - M[I] \tag{4}$$

where $M[P]$, $M[E]$ and $M[I]$ are the mathematical expectations of the precipitation, evaporation and infiltration rates, respectively.

Assume that the mathematical expectation of the input values (precipitation, air moisture deficit and wind velocity) is equal to their mean values obtained on the basis of the measurements. Assume also that the distribution of the maximum moisture storage capacities over the catchment area (distribution of relative areas with different maximum moisture storage values) can be written as an exponential relation

$$\phi(W_m) = \frac{1}{k} \exp\left(- \frac{W_m}{k}\right) \tag{5}$$

where k is a distribution parameter. If the mathematical expectation of the maximum moisture storage capacities is $M[W_m]$ it is easy to show that $k = M[W_m]$.

Using equation (5) it is possible to find the probability that at each point the soil moisture deficit does not exceed W_m:

$$P(d_i < W_m) = \int_{d_i}^{\infty} \phi(W_m)\, dW_m = \exp\left(- \frac{d_i}{M[W_m]}\right) \tag{6}$$

If the soil moisture deficit throughout the catchment area is assumed to be equal to its mathematical expectation $M[d]$ the mathematical expectation of evaporation can be found by the relation:

$$M[E] = \{k_1 M[D] + k_2 M[D]M[U] + K_{DU}\} \exp\left(-\frac{M[d]}{M[W_m]}\right) \tag{7}$$

where K_{DU} is the covariance between air moisture and wind velocity.

Taking the mathematical expectation in equation (3) gives

$$M[I] = \left[\frac{M[d]}{k_3} + i_0\right] \int_{M[I]}^{\infty} f_2(P-E)\, d(P-E)$$

$$+ M[P-E] \int_0^{M[I]} f_2(P-E)\, d(P-E) \tag{8}$$

where $f_2(P-E)$ is the probability distribution of $(P-E)$.

Since the evaporation during precipitation is negligible we can take $f_2(P-E) = f_2(P)$ in equation (8). Distribution of $f_2(P)$ can be considered to be normal.

If the calculation is started from a certain time when the moisture deficit can be considered to equal zero, then

$$M[d] = M[W_m] - \int_0^t \{M[E] + M[Q] + M[I] - M[P]\}\, d\tau \tag{9}$$

where Q is the runoff relative depth.

In order to pass from the water yield to the surface runoff, introduce the distribution of the active catchment area. Assume that plot r of the area is undrained and the increment of active catchment area is proportional to the surface runoff depth. Then, the surface runoff is:

$$q_s(t) = M[q]r\left[1 - \exp\left(-m \int_0^t M[q]\, d\tau\right)\right] \tag{10}$$

where r and m are empirical parameters.

The subsurface runoff can be taken into account by the formula

$$q_I(t) = i_0 \exp(-k_4 M[d]) \tag{11}$$

where k_4 is an empirical parameter.

Using the linear model to route the surface and subsurface runoff gives

$$Q(t) = \int_0^t P_1(t-\tau)q_s(\tau)\, d\tau + \int_0^t P_2(t-\tau)q_I(\tau)\, d\tau \tag{12}$$

where $P_1(t)$ and $P_2(t)$ are transfer functions.

As practice shows [the use of Kalinin–Milyukov's method (1958) and Nash's model (1960)] the transfer function may be approximated by the gamma distribution therefore it is possible to take

$$P_i(t) = \frac{1}{\tau_i \Gamma(n_i)}\left(\frac{t}{\tau_i}\right)^{n_i-1} \exp(-t/\tau_i) \qquad (i = 1, 2) \tag{13}$$

where τ_i, n_i are empirical parameters.

The described model has been taken as the basis for constructing the model of runoff formation in several catchments of areas up to $15{,}000\ \text{km}^2$. The model si...
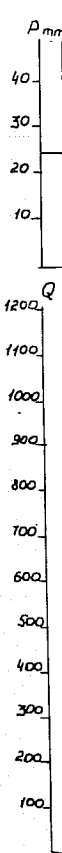
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is gradually complicated by including new elementary processes and parameters.
At the beginning the model was tested containing only the r parameter which in this
case can be considered as a runoff coefficient, and the parameters of the surface runoff
transfer function, τ_1 and n_1. Then, the parameters accounting for the effect of antecedent
soil moisture (W_m, k_1, k_3) were included. Further, the model was made more precise by
a more accurate account of the active catchment area (parameter m), more accurate
account of infiltration (parameter i_0), account of subsurface runoff (parameters k_5, τ_2,
n_2), and, finally, by a more accurate account of evaporation (parameter k_2). The model
was continually complicated as long as a noticeable improvement was observed in the
agreement between the actual and calculated hydrographs both for the flood events
used to calibrate the model and for the flood events used as controls. To illustrate the
procedure of making the model more precise Fig. 1 shows the comparison of the actual
runoff hydrograph and hydrographs obtained with the aid of models of different
complexity. As the model is complicated the determination of the parameters is
complicated as well. For the simplest model the parameters r, τ_1 and n_1 were
analytically determined from the observed precipitation and hydrographs. Optimization
methods were used for more complicated models. The application of optimization
methods to determine the parameters of the runoff models shows that the rate of
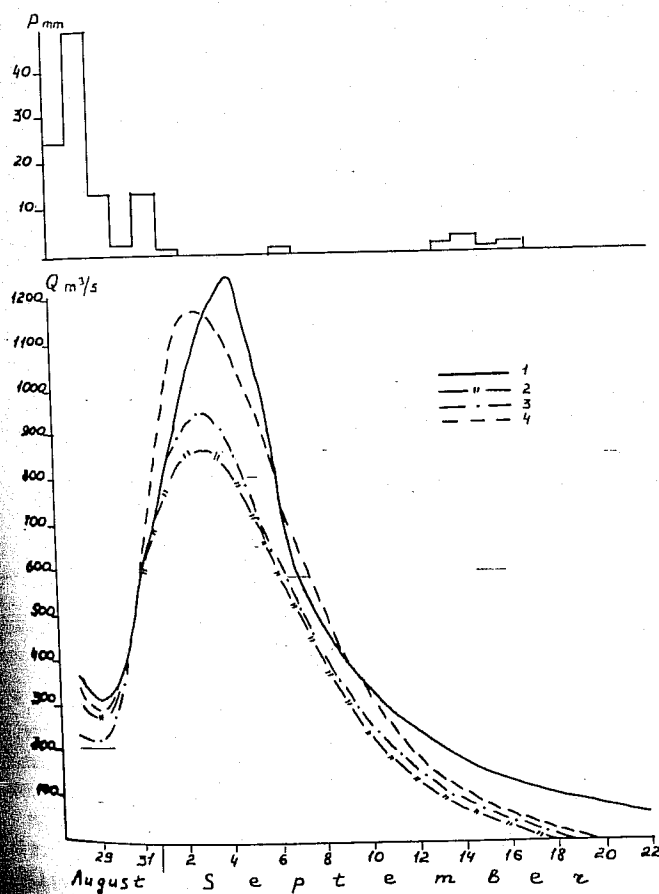convergence of the parameters to their optimum values depends strongly on the initial

FIGURE 1.   Actual hydrograph (1) and hydrographs calculated for control using models
with parameters: (4) τ_1, n_1 η, m; (3) τ_1, n_1, η, $1/k_3$, W_m, k_1; (2) τ_1, n_1, η, i_0, $1/k_3$, W_m, k_1, k_5,
River Bikin-V. Olon, 1966.

approximation. *A priori* it is difficult to preset a fitting criterion which the optimization must achieve. The absolute minimum value either cannot be achieved at all with the initial data available or it requires too much computer time. The selection of good initial approximation of the parameters along with reasonable constraints on their values obtained from physical concepts creates a reliable basis for optimization. The successive model complication makes it possible to use as an initial approximation the values of the corresponding parameters from a simpler model. On introduction of the next parameter all the parameters are optimized anew.

The structure of the fitting criterion is of great importance in determining the parameters by optimization methods. A correct selection of this structure permits, on the one hand, an evaluation of how the basic physical laws (conservation laws) are met in the models, and on the other hand consideration of the different uses of the available data under the conditions of a specific problem. In order that the parameter optimization take into account the diverse conditions in which each of the parameters can play a considerable part it is desirable to calculate the fitting criterion on as many of the recorded floods as possible. However, the speed of modern computers makes it possible to use only a few floods in optimizing the parameters of complicated models (we have used four to five floods). Therefore, it is necessary that the fitting criterion includes, if possible, a great amount of information about each hydrograph even though it is achieved at the expense of decreased accuracy of determination of some parameters. Thus, for instance, the fitting criterion calculated as a mean-root-square (standard) deviation of the calculated and actual volumes appears inapplicable to sufficiently complicated models in spite of the fact that it allows the best evaluation of the parameters which affect the runoff volume. This criterion results in a sharp reduction of the information contained in the recorded hydrograph; all the ordinates of the hydrograph are replaced by one value. However, an effort to make the fitting criterion informative to the maximum extent can result in its increased sensitivity to the errors found in the initial data. This, for example, is observed if the fitting criterion takes into consideration the hydrograph time derivatives.

We have used in the main the following two fitting criteria:

$$K_1 = \sum_{j=1}^{n} \frac{1}{V_j} \left\{ \int_0^{T_j} [Q_{\phi j}(\tau) - Q_{pj}(\tau)]^2 \, d\tau \right\}^{1/2} \tag{14}$$

$$K_2 = \sum_{j=1}^{n} \frac{1}{V_j} \left\{ \int_0^{T_j} [Q_{\phi j}(\tau) - Q_{pj}(\tau)]^2 Q_{\phi j}(\tau) \, d\tau \right\}^{1/2} \tag{15}$$

where

$Q_{\phi j}$ and Q_{pj} are actual and calculated ordinates of the j-hydrograph;
T_j is its duration;
x_i is the vector of parameters;
V_j is the runoff volume;
n is the number of hydrographs.

The first yields rather smooth hydrographs due to the errors for high and low discharges being similarly treated. The second criterion mainly attaches importance to large discharges; however, it gives less consideration to the maintenance of volumes.

In selecting an optimization method we tried to use a method which operates well with both a large number of parameters and a complicated response surface of the objective function. Such a procedure has been developed at the Computer Centre of Moscow University to solve problems in the field of chemical technology (Polyak and Skokov, 1967). This procedure can be used for successive minimization of the fitting criterion by the combined application of the univariate method with the squ

approximation with respect to each variable and the conjugate gradient method. The penalty function method was used to consider the limitations on the parameters.

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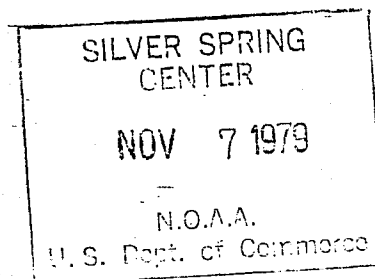
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